08/8/2020

B. Sc. Part I 1st Paper

$\boxed{\text{TRIGONOMETRY}}$

Summation of series (contd.)

1. Sum the following series

$$r\cos\alpha - \frac{r^2}{2}\cos 2\alpha + \frac{r^3}{3}\cos 3\alpha \cdots \text{ to } \infty$$

Soln.

Let $C = r\cos\alpha - \frac{r^2}{2}\cos 2\alpha + \frac{r^3}{3}\cos 3\alpha \cdots \text{ to } \infty$.

Let $S = r\sin\alpha - \frac{r^2}{2}\sin 2\alpha + \frac{r^3}{3}\sin 3\alpha \cdots \text{ to } \infty$.

$$\Rightarrow C+iS = k(\cos\alpha + i\sin\alpha) - \frac{k^2}{2}(\cos 2\alpha + i\sin 2\alpha) + \frac{k^3}{3}(\cos 3\alpha + i\sin 3\alpha) \cdots \text{ to } \infty$$

$$\Rightarrow C+iS = k e^{i\alpha} - \frac{k^2}{2} e^{2i\alpha} + \frac{k^3}{3} e^{3i\alpha} \cdots \text{ to } \infty$$

Put $k e^{i\alpha} = x$

$$\Rightarrow C+iS = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \text{ to } \infty$$

$$\Rightarrow C+iS = \log(1+x) = \log(1+ke^{i\alpha})$$

$$\Rightarrow C+iS = \log\{1 + k\cos\alpha + ik\sin\alpha\}$$

$$\Rightarrow C+iS = \log\{(1+k\cos\alpha) + ik\sin\alpha\}$$

$$\Rightarrow C+iS = \frac{1}{2}\log\{(1+k\cos\alpha)^2 + k^2\sin^2\alpha\} + i\tan^{-1}\frac{k\sin\alpha}{1+k\cos\alpha}$$

$\Rightarrow C+iS = \frac{1}{2}\log\{1+2k\cos\alpha+k^2\} + i\tan^{-1}\frac{k\sin\alpha}{1+k\cos\alpha}$

Equating real parts, we get

$$C = \frac{1}{2}\log\{1+2k\cos\alpha+k^2\},$$

Q. Sum the series

(a) $\cos\alpha - \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha - \cdots$ to ∞

(b) $\sin\alpha - \frac{1}{2}\sin 2\alpha + \frac{1}{3}\sin 3\alpha - \cdots$ to ∞

Soln. Let $C = \cos\alpha - \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha - \cdots$ to ∞

and $S = \sin\alpha - \frac{1}{2}\sin 2\alpha + \frac{1}{3}\sin 3\alpha - \cdots$ to ∞

$\Rightarrow C+iS = (\cos\alpha + i\sin\alpha) - \frac{1}{2}(\cos 2\alpha + i\sin 2\alpha)$
$+ \frac{1}{3}(\cos 3\alpha + i\sin 3\alpha) - \cdots$ to ∞

$\Rightarrow C+iS = e^{i\alpha} - \frac{1}{2}e^{2i\alpha} + \frac{1}{3}e^{3i\alpha} - \cdots$ to ∞

Put $e^{i\alpha} = x$

$\Rightarrow C+iS = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ to ∞

$\Rightarrow C+iS = \log(1+x) = \log(1+e^{i\alpha})$

$\Rightarrow C+iS = \log(1+\cos\alpha + i\sin\alpha)$

$\Rightarrow C+iS = \frac{1}{2}\log\{(1+\cos\alpha)^2 + \sin^2\alpha\} + i\tan^{-1}\frac{\sin\alpha}{1+\cos\alpha}$

$$\Rightarrow c+is = \frac{1}{2}\log\{1+\cos^2\alpha+2\cos\alpha+\sin^2\alpha\}$$

$$+ i\tan^{-1}\frac{\cancel{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cancel{2}\cos^2\frac{\alpha}{2}}$$

$$\Rightarrow c+is = \frac{1}{2}\log\{1+1+2\cos\alpha\} + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\Rightarrow c+is = \frac{1}{2}\log\{2(1+\cos\alpha)\} + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\Rightarrow c+is = \frac{1}{2}\log\left\{2\times 2\cos^2\frac{\alpha}{2}\right\} + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\Rightarrow c+is = \frac{1}{2}\log\left\{\left(2\cos\frac{\alpha}{2}\right)^2\right\} + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\Rightarrow c+is = \frac{1}{2}\times 2\log\left(2\cos\frac{\alpha}{2}\right) + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\Rightarrow c+is = \log\left(2\cos\frac{\alpha}{2}\right) + i\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

Equating real and imaginary parts, we get

$$c = \log\left(2\cos\frac{\alpha}{2}\right)$$

and $s = \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \frac{\alpha}{2}$ except when $\alpha = (2n+1)\pi$